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Topology of Generic Multijet Preimages and Blow-Up via Newton Interpolation

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We study the local topological structure of generic multijet preimages of algebraic varieties and prove their stratifiability with additional quantitative estimates.

The proof is based on interpretation of Newton interpolation formula as a resolution of singularities on the diagonal of the multijet space.

1. BIFURCATION OF LIMIT CYCLES AND EQUATIONS INVOLVING MULTIJETS

1.1. *Bifurcations of Planar Separatrix Polygons and Mixed Systems of Equations*

The object of study in this paper is the structure of loci defined by mixed systems of polynomial and finitely-smooth equations. More specifically, we need to describe the structure of a common zero locus in $\mathbb{R}^n$ of a finite number of functions, each one of them being the composition of a polynomial and a smooth function. The description should be based on a very meager data. The only information available on the polynomials is an upper bound for their degrees, and the smooth functions are simply assumed to be generic.

This seemingly artificial problem in fact constitutes one of the key steps in the investigation of bifurcation of limit cycles from separatrix polygons (polycycles) [IY]. This lengthy paper contains an explicit upper bound for the number of limit cycles that can be born from a separatrix polygon carrying only elementary singularities, in a generic p-parametric family. The answer was given by a primitive recursive function $E(p)$ depending only on the number of parameters (the degeneracy codimension of the polygon and the family).

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The result on the structure of the loci, that was used to produce the above estimate, looks as follows. Let $P_1, ..., P_N$ be real polynomials of degree $\leqslant d$ defined on the m-jet space $J^m(\mathbb{R}^s, \mathbb{R}^k)$ of vector functions of n independent variables, and $j^m f: \mathbb{R}^s \to J^m(\mathbb{R}^s, \mathbb{R}^k)$ is an m-jet extension of a smooth vector function $f: \mathbb{R}^s \to \mathbb{R}^k$. Then the locus

$$\{x: P_\alpha(j^m f(x)) = 0, \alpha = 1, ..., N\} \subset \mathbb{R}^s \tag{1}$$

for a generic smooth function f is a stratified subvariety in $\mathbb{R}^s$, and the local topological complexity of this subvariety (see below) admits an upper estimate by an explicit function of d, s, m, k. Here the genericity means that the assertion is valid for all functions from a certain (in general, depending on P_α) residual subset in the space of sufficiently smooth functions.

The equations of such form naturally appear after elimination of Pfaffian equations from the system describing fixed points of the Poincaré return map of the polycycle, see [IY]: the polynomial part ultimately originates in the integrable normal forms around elementary singular points, and the generic smooth functions f_j correspond to flow maps along the sides of the separatrix polygon.

The proof of stratifiability of the locus (1) is one line long: this locus is the preimage of a real algebraic variety $\Sigma = \{P_\alpha = 0\} \subset J^m(\mathbb{R}^s, \mathbb{R}^k)$. By the classical Whitney theorem [W], Σ is a stratified variety and its local topological complexity can be bounded from above in terms of the degrees of the polynomials P_α. Now from the strong Thom's transversality theorem [GG] one can derive that the set of functions whose m-jets are transversal to Σ is residual. It remains only to remark that a transversal preimage of a stratified variety will again be a stratified variety, and the local topological complexity cannot increase when taking such preimages. Notice in particular, that if Σ were smooth, then the preimage would be also a smooth variety.

1.2. *Spatial Polycycles and Multijets*

The above approach can be almost verbatim applied to bifurcations of *spatial* polygons, i.e., closed piecewise-smooth curves in $\mathbb{R}^\ell$, $\ell > 2$, consisting of singular points connected by heteroclinic orbits of a vector field in the space.

However, there are two circumstances to be taken into consideration. First, the correspondence maps near singular points may be non-Pfaffian, even for the linearizable singularities: usually this occurs when the spectrum contains non-real eigenvalues. A typical example is the Shilnikov loop of a saddle-focus (1 positive real eigenvalue and two complex conjugate eigenvalues with the negative real part). In this case there can occur an infinite number of closed orbits accumulating to the polycycle. This situation needs to be explicitly excluded from considerations.

Besides this natural restriction on the types of singularities (only those with Pfaffian correspondence maps can be allowed), the second new phenomenon is the occurrence of n-periodic orbits (those that close up after $n \geqslant 1$ "turns" around the polygon): on the plane the situation with $n > 1$ is impossible for topological reasons. Actually, even without perturbing the vector field one may generically have an *infinite* number of n-periodic orbits accumulating to the polycycle, with arbitrarily large periods.

What one can expect (at least in the case when all singularities have Pfaffian correspondence maps), is that for any finite n the number of n-periodic orbits born from the polycycle will be bounded by an explicit function of n and other relevant natural parameters (the dimension of the phase space and the number of parameters of the family).

This conjecture can be indeed proved (even with a better than in [IY] upper bound) using the same tools as in the planar case, as shown recently by V. Kaloshin [K]. However, this proof requires replacing the system of equations describing the locus as in (1), by a *multijet* system of the form

$$\{(x_1, ..., x_n): P_\alpha(j^m f(x_1), ..., j^m f(x_n)) = 0\}_{\alpha=1}^{N} \subset \underbrace{\mathbb{R}^s \times \cdots \times \mathbb{R}^s}_{n \text{ times}}. \tag{2}$$

As before, the question to be answered is on the structure of the locus (2) and its local topological complexity in terms of the integer data, specifically, n, s, m, k, d. The problem is rooted in the fact that *the multijet transversality theorem does not hold on the diagonal*, as the points x_j tend to each other [GG].

1.3. *Definitions, Preliminary Results and Formulation of the Problem*

The main problem can be formulated as follows: prove that for an arbitrary algebraic subvariety Σ of a multijet space, its preimage by a multijet extension of a generic smooth map is a stratified subvariety. In addition, the local topological complexity of this preimage, measured by its *contiguity number* (see below), is to be bounded in terms of the dimensions, number of repetitions and degrees of equations determining Σ.

The accurate definitions follow. A *stratified subvariety* of a Euclidean space is a locally finite collection of smooth submanifolds of various dimensions (*strata*), such that the closure of each stratum is the union of the stratum itself and some of the strata of inferior dimensions. In addition, it is required that the tangent spaces at interior points of strata exhibit certain regular limit behavior when approaching the boundary (the so called *Whitney condition* "*B*"). Instead of formulating this condition explicitly, we mention some important corollaries: first, all real algebraic subvarieties can be *stratified* (represented as unions of strata as above), as proved by Whitney [W]. Second, the property of a smooth map to be transversal to a compact stratified variety (i.e., to be transversal to all its strata) is an open property. Third, the transversal preimages of stratified varieties are themselves stratified varieties (both the two properties can be found in [M]).

The *contiguity number* is an integer index introduced in [IY] as a degree of local topological complexity. It is first defined for one-dimensional stratified varieties as the maximal number of edges (1-dimensional strata) that can land on each single vertex (0-dimensional stratum).

For an arbitrary stratified subvariety V of codimension c (i.e., the union of smooth strata of codimenions c and more), the intersection with a generic $(c+1)$-dimensional smooth submanifold M (transversal to the V) will be 1-dimensional stratified variety. By definition, the contiguity number of M is the supremum of the contiguity numbers of $M \cap V$ taken over all $(c+1)$-dimensional manifolds M transversal to V. Obviously, if V were a smooth variety arbitrarily partitioned into strata subject to the above requirements, then the contiguity number of V can be at most 2. Hence any larger value of the contiguity number singnals a nontrivial local topological structure of V. Somewhat naively, the upper-dimensional strata of V can be thought of as pages of a book sticking together along the strata of inferior dimensions: the contiguity number is the maximal number of such pages that stick to each other at a generic point.

It can be easily shown from definitions that *the contiguity number of a transversal preimage of a stratified variety cannot exceed that of the variety itself.*

On the other hand, one can use Bézout theorem in the form due to J. Heintz [H] to place an upper bound for the contiguity number of a real algebraic variety: *For a real algebraic locus $\Sigma \subset \mathbb{R}^s$ defined by polynomial equations of degree $\leqslant d$, the contiguity number does not exceed* $2d^{s-1}$ (a slightly weaker inequality was derived in [IY] from the Milnor's estimate for Betti numbers).

1.4. *Instructive Example*

The problem formulated above is not merely a technical question on regularity of certain loci. The appearance of stratified varieties that are not smooth manifolds can sometimes look intrigueing.

Consider the simplest case of $m=0$ (0-jets), $s=1$ (functions of one variable) and $q=2$ (only one repetition). The multijet space is 4-dimensional with the coordinates (x_1, x_2, v_1, v_2). Let Σ be a hyperplane given by the equation $\lambda_1 v_1 + \lambda_2 v_2 = \lambda_0$.

One can show relatively easily, using the standard multiject transversality theorem [GG], that if $(\lambda_0 : \lambda_1 : \lambda_2) \neq (0 : 1 : -1)$, then the typical preimage $\{(x_1, x_2) \in \mathbb{R}^2 \colon \lambda_1 f(x_1) + \lambda_2 f(x_2) = \lambda_0\}$ will be a smooth curve.

However, the locus $\{f(x_1)=f(x_2)\}\subset\mathbb{R}^2$ is never a smooth curve. Yet for a generic (actually, for a Morse) function $f\colon\mathbb{R}^1\to\mathbb{R}^1$ the above locus will be a smooth curve with normal crossing, i.e., a stratified one-dimensional variety with the contiguity number equal to 4.

This example shows that even in the simplest settings the stratified varieties appear naturally as multijet preimages. Our goal was to prove this under the most general assumptions and, in particular, to explain why in the above example one case is so much exceptional.

1.5. *Main Result and the Idea of the Proof*

A generic multijet preimage of an algebraic variety is a stratified subvariety. The first proof of this result was achieved in [G]: here we give a simplified version.

THEOREM (Main). *For any real algebraic variety $\Sigma\subseteq J^m(\mathbb{R}^s,\mathbb{R}^k)\times\cdots\times J^m(\mathbb{R}^s,\mathbb{R}^k)$ (n times) and any bounded ball $B\subset\mathbb{R}^s\times\cdots\times\mathbb{R}^s$ (n times) there exists an open dense subset $\mathscr{U}=\mathscr{U}_{\Sigma,B}\subset C^N(\mathbb{R}^k,\mathbb{R}^s)$ in the space of sufficiently smooth functions, such that for any $f\in\mathscr{U}$ the multijet preimage $(j^mf\times\cdots\times j^mf)^{-1}(\Sigma)\subset\mathbb{R}^{s\times n}$ will be a stratified variety inside B.*

If the set Σ is defined by polynomial equations of degree $\leqslant d$, then the contiguity number of the generic preimage does not exceed $2(n(m+1)\,sd)^{ns+k\cdot(n(m+1))^s}$.

The proof of this result is obtained by blowing up the diagonal of the multijet space $\mathbf{J}(n,m,s,k)=J^m(\mathbb{R}^s,\mathbb{R}^k)\times\cdots\times J^m(\mathbb{R}^s,\mathbb{R}^k)$ (n times). We construct an appropriate space $\mathbf{D}=\mathbf{D}(n,m,s,k)$ (depending on the integer parameters of the multijet space) together with a polynomial projection $\pi\colon\mathbf{D}(n,m,s,k)\to\mathbf{J}(n,m,s,k)$ in such a way that the following two properties hold:

(1) for any sufficiently smooth map $f\colon\mathbb{R}^s\to\mathbb{R}^k$ its multijet extension map $\mathscr{J}^{m,n}f=j^mf\times\cdots\times j^mf\colon\mathbb{R}^s\times\cdots\times\mathbb{R}^s\to\mathbf{J}(n,m,s,k)$ is "covered" by a smooth map $\mathscr{D}^{n,m}f\colon\mathbb{R}^s\times\cdots\times\mathbb{R}^s\to\mathbf{D}(n,m,s,k)$ in the sense that $\mathscr{J}^{m,n}f=\pi\circ\mathscr{D}^{n,m}f$, and

(2) by an arbitrarily C^∞-small variation of f one can achieve the transversality of $\mathscr{D}^{n,m}f$ to any stratified subvariety in the target space $\mathbf{D}(n,m,s,k)$.

Having constructed the blow-up π actually solves the problem, since the blow-up of any algebraic variety Σ in the multijet space, will be an algebraic (hence stratifiable) subvariety S in the resolution space. The second condition guarantees that the transversality of $\mathscr{D}^{n,m}f$ to S is an open dense property, and for such generic f we immediately conclude with stratifiability of $(\mathscr{D}^{n,m}f)^{-1}(S)=(\mathscr{J}^{n,m}f)^{-1}(\Sigma)$.

One should look for a “well balanced” construction, since “overblowing” might easily destroy the first condition, while “underblowing” may be insufficient to ensure the second one. The rest of the paper is concerned with constructing explicitly the desingularization π. The compromise is achieved by using the Newton interpolation formula with unequal intervals and *divided differences*. The above two properties of the blow-up π that is constructed below, follow from the corresponding two properties of the divided differences:

(1) divided differences of a smooth function remain smooth functions of the points on the interpolation grid, even as some points of this grid eventually coalesce, and

(2) the map taking the linear space of all polynomials of sufficiently high degree into the collection of all divided differences, is surjective.

The following section contains the necessary background from the interpolation theory. The concluding Section 3 repeats the above argument with more details. We conclude with discussion of the example given in Section 1.4 and unveil the mystery.

2. DIVIDED DIFFERENCES AND NEWTON INTERPOLATION FORMULA

2.1. *Language of Divided Differences*

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a smooth function of n real variables $x_1, ..., x_n$. The *first order divided difference* of f in the variable x_k is the function of $n+1$ variables $x_1, ..., x_{k-1}, x'_k, x''_k, x_{k+1}, ..., x_n$, defined as

$$\mathbf{\Delta}_{x_k} f(x_1, ..., x_{k-1}, x'_k, x''_k, x_{k+1}, ..., x_n) = \frac{f(x_1, ..., x_{k-1}, x'_k, x_{k+1}, ..., x_n) - f(x_1, ..., x_{k-1}, x''_k, x_{k+1}, ..., x_n)}{x'_k - x''_k} \tag{3}$$

for $x'_k \neq x''_k$ and extended by its limit value as $(\partial f/\partial x_k)(x_1, ..., x_k, ..., x_n)$ for $x'_k = x''_k = x_k$. Clearly, (e.g., by the Hadamard lemma), $\mathbf{\Delta}_{x_k} f$ is at least C^{r-1}-smooth function of its arguments, if f was C^r-smooth.

Iterating this construction is therefore possible, giving rise to *divided differences of nth order* for all $n \geqslant 2$. This could lead to somewhat awkward notation, since formally the operation $\mathbf{\Delta}_{x_k} \mathbf{\Delta}_{x_k}$ makes no sense: one should decide between $\mathbf{\Delta}_{x'_k} \mathbf{\Delta}_{x_k}$ and $\mathbf{\Delta}_{x''_k} \mathbf{\Delta}_{x_k}$. Fortunately, the result will be the same, as an easy computation shows [BZ]. Moreover, it is clear that the operators $\mathbf{\Delta}_{x_k}$ and $\mathbf{\Delta}_{x_j}$ commute for $k \neq j$, and therefore we can use unambiguously the multi-index notation: for $\alpha=(\alpha_1, ..., \alpha_n)\in\mathbb{Z}^n_+$ and a function $f(x)$ of n variables $x=(x_1, ..., x_n)$ we denote $\mathbf{\Delta}^\alpha_x f=\mathbf{\Delta}^{\alpha_1}_{x_1}\cdots\mathbf{\Delta}^{\alpha_n}_{x_n} f$ the mixed divided difference of order $|\alpha|=\alpha_1+\cdots+\alpha_n$. Notice that this divided difference is a smooth function of $n+|\alpha|$ arguments subdivided into n groups of $\alpha_1+1, ..., \alpha_n+1$ variables, symmetric with respect to permutations of variables *within the same groups*.

2.2. *Newton Interpolation Formula in one Variable*

The divided differences occur as coefficients of interpolating polynomials, and this can be considered as their characteristic property. More precisely, if $f=f(x)$ is a function of *one* variable (and therefore the subscripts of the difference operators may be omitted), then

$$\begin{aligned} f(t)=\mathbf{\Delta}^0 f(x_1)+\mathbf{\Delta}^1 f(x_1, x_2)\cdot(t-x_1)+\cdots \\ +\mathbf{\Delta}^{n-1} f(x_1, ..., x_n)\cdot(t-x_1)\cdots(t-x_{n-1}) \\ +\mathbf{\Delta}^n f(x_1, ..., x_n, t)\cdot(t-x_1)\cdots(t-x_{n-1})(t-x_n) \end{aligned}\tag{4}$$

identically for all values of $t, x_1, ..., x_n$. All terms of this representation, except for the last one, are polynomial in t and their sum is the nth order *Newton interpolation polynomial* denoted by $\mathscr{P}_{n-1}(t; X)$, where $X=(x_1, ..., x_n)$. The smooth *remainder* term $R_n(t, X)=f(t)-\mathscr{P}_{n-1}(t; X)=\mathbf{\Delta}^n f(x_1, ..., x_n, t)$ $\cdot(t-x_1)\cdots(t-x_{n-1})(t-x_n)$ vanishes for $t=x_1, ..., x_n$, and therefore we have the following basic fact: *For any collection of points $X=\{x_1, ..., x_n\}$ on the real line, the values $f(x_k)$ at all points of X can be restored by a polynomial formula from the coordinates of these points and the divided differences $u_\alpha=\mathbf{\Delta}^\alpha f(x_1, ..., x_{\alpha+1})$ of all orders $\alpha=0, ..., n-1$.* The degree of this polynomial is $\leqslant n$ in all variables $x_1, ..., x_n, u_0, ..., u_{\alpha-1}$.

The above formula remains stable as the points x_i eventually coalesce. This is the principal difference between the Newton and the Lagrange interpolation formulas: the Lagrange polynomials $L_i(t)$ that are defined by the characteristic property $L_i(x_j)=\delta_{ij}$ (the Kronecker delta), do not have any regular limit as $x_j\to x_i$.

Suppose now that the set $\tilde{X}$ contains each point repeated $m+1$ times:

$$\tilde{X}=\{\underbrace{x_1, ..., x_1}_{m+1 \text{ times}}, \underbrace{x_2, ..., x_2}_{m+1 \text{ times}}, ..., \underbrace{x_n, ..., x_n}_{m+1 \text{ times}}\}, \qquad x_k\in\mathbb{R}, \quad x_k\neq x_j \quad \text{for } k\neq j. \tag{5}$$

Then one can evaluate all divided differences of order $\leqslant(m+1)\,n-1$ using the first $\alpha+1$ points of $\tilde{X}$ as arguments for $\mathbf{\Delta}^\alpha f$, and construct the interpolating polynomial of order $(m+1)\,n-1$ as in (4). But the residual term $R_{n(m+1)}(t, \tilde{X})$ will be then a function that has zero m-jet at all points x_j, and therefore the m-jet $j^m f(x_j)$ is completely determined by the Newton polynomial $\mathscr{P}_{n(m+1)-1}(t, \tilde{X})$ and its derivatives in t. The formula restoring each derivative is polynomial of degree $\leqslant (m+1)n$ in x_j and $u_\alpha = \mathbf{\Delta}^\alpha f(\tilde{X})$, $\alpha = 0, ..., n(m+1)-1$.

2.3. *Blow-up of the Multijet Space for Functions of one Variable*

We describe here the polynomial map desingularizing the multijet space in the simplest case of functions of one real variable. This is done by introducing a proper formalism in the constructions of the preceding section.

DEFINITION. The multijet (or, more explicitly, an n-multi-m-jet) of a smooth vector valued function $f: \mathbb{R}^s \to \mathbb{R}^k$ is a map $\mathscr{J}^{m,n} f$ that takes an ordered collection of n points $(x_1, ..., x_n) \in (\mathbb{R}^s)^n$ into the ordered collection $(j^m f(x_1), ..., j^m f(x_n))$ of m-jets at the corresponding points. The right hand side $(J^m(\mathbb{R}^s, \mathbb{R}^k))^n$ is called the multijet space and denoted $\mathbf{J}(n, m, s, k)$.

DEFINITION. Let $f: \mathbb{R}^1 \to \mathbb{R}^1$ be a smooth function of one variable. Its *DD-extension* (divided difference extension) of order n is a map $\mathscr{D}^n f: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, defined as

$$(x_1, ..., x_{n+1}) \mapsto (x_1, ..., x_{n+1}, u_0, ..., u_n),$$
$$u_\alpha = \mathbf{\Delta}^\alpha f(x_1, ..., x_{\alpha+1}), \qquad \alpha = 0, ..., n.$$

The target space of the DD-extension is a discrete analog of the multijet space, that will be referred to as the divided difference space (more explicitly, nth order divided difference space) and denoted, by analogy with the multijet space, by $\mathbf{D}(n, 0, 1, 1)$.

Denote by $diag^m: \mathbb{R}^n \to \mathbb{R}^{(m+1)n}$ the map that takes the ordered collection of points $X = (x_1, ..., x_n)$ into the ordered collection $\tilde{X}$ as in (5), where each point is repeated $m+1$ times.

DEFINITION. The *DD-extension with m repetitions* of a function f of one variable is the composition of the m-diagonal map diag^m with the DD-extension of f of order $(m+1)n$, which means that the point $(x_1, ..., x_n)$ is taken into all divided differences of all possible orders, evaluated on the set $\operatorname{diag}^m(X)$ which contains the points $(x_1, ..., x_n)$ each repeated $m+1$ times:

$$\mathscr{D}^{n,m} f(X) = \mathscr{D}^{n(m+1)} f(diag^m(X)), \qquad X = (x_1, ..., x_n).$$

The obvious notation for the target space of the DD-extension with m repetitions is $\mathbf{D}(n, m, 1, 1)$ (its multivariate analog $\mathbf{D}(n, m, s, k)$ will be introduced later).

In other words, we define the DD-extension $\mathscr{D}^{n,m}$ of order n with m repetitions as the restriction of the (simple) DD-extension of order

$n(m+1)$ (without repetitions) on the "m-diagonal" of the source space (the coordinates are partitioned into n equal groups and set equal to the same value x_k within each group independently).

The interpolation formula (4) together with its formal derivatives of orders $\leqslant m$ evaluated at $t = x_1, ..., x_n$, restoring the multijet from the collection of the divided differences, now can be interpreted as a polynomial map between the two spaces.

Newton Interpolation on $\mathbb{R}^1$ (Abstract Version). *The Newton interpolation formula* (4) *for smooth functions of one variable defines a polynomial map* π *of the divided difference space* $\mathbf{D}(n, m, 1, 1)$ *into the corresponding multijet space* $\mathbf{J}(n, m, 1, 1)$ *in such a way that*

$$\mathscr{J}^{m, n} f = \pi \circ \mathscr{D}^{n, m} f.$$

All coordinates of the map π *are polynomials of degree* $\leqslant n(m+1)$. (*See Fig.* 1.)

In the simplest settings when repetitions are not allowed $(m=0)$, the n-multi-0-jet space is equipped with the coordinates $(x_1, ..., x_n, v_1, ..., v_n)$, v_j being the "value at the point x_j." The map π in these coordinates takes the form (the trivial equations $x_j = x_j$, $j = 1, ..., n$, tacitly assumed)

$$\begin{aligned}
v_1 &= u_0, \\
v_2 &= u_0 + u_1(x_2 - x_1), \\
v_3 &= u_0 + u_1(x_3 - x_1) + u_2(x_3 - x_1)(x_3 - x_2), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
v_n &= u_0 + u_1(x_n - x_1) + \cdots + u_{n-1}(x_n - x_1) \cdots (x_n - x_{n-1}).
\end{aligned}$$

This map is invertible outside the union of codimension 2 subspaces $\{x_i = x_j,\ v_i = v_j\}$, as the values u_α are uniquely restored using the definition (3) in the complement to this union.

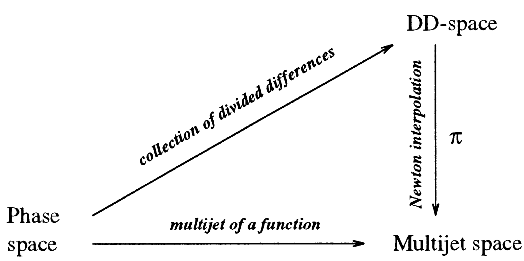

FIG. 1. Blow-up via Newton interpolation formula.

2.4. *Interpolation of Multivariate Functions*

Newton interpolation of functions of several variables is somewhat more tricky, see the discussion in [BZ]. However, since we are not restricted by optimality consideration, an excessive construction can be utilized.

Note that the Newton interpolation formula (4) can be written for a function f which depends smoothly on some additional parameters. The coefficients (the divided differences) will then be smooth functions of these parameters as well. This allows recursive application of the formula (4) to constructing multivariate interpolation polynomials.

Let $X^1 = (x_1^1, ..., x_n^1) \subset \mathbb{R}, ..., X^s = (x_1^s, ..., x_n^s) \subset \mathbb{R}$ be s subsets consisting of the same number of points, each X^j belonging to the corresponding jth cordinate axis of the space $\mathbb{R}^s$. Then, given a multiindex $\alpha \in \mathbb{Z}_+^s$ and a function of s real variables $f(x) = f(x^1, ..., x^s)$, we can form the divided differences $\mathbf{\Delta}_x^\alpha f(X^1, ..., X^s)$. (To save on the length of formulas, we will always assume that if the divided difference of some multiorder α is evaluated on a collection of ordered sets such that jth set contains more than $\alpha_j + 1$ points, then only the first $\alpha_j + 1$ of them are taken.)

In terms of the divided differences one can write the Newton interpolating polynomial as follows:

$$\mathscr{P}(t^1, ..., t^s)$$

$$= \sum_{0 \leqslant \alpha_i \leqslant n-1} \mathbf{\Delta}_x^\alpha f(X^1, ..., X^s) \cdot \prod_{i_1=1}^{\alpha_1} \cdots \prod_{i_s=1}^{\alpha_s} (t^1 - x_{i_1}^1) \cdots (t^s - x_{i_s}^s). \tag{6}$$

This cumbersome expression is a polynomial of degree $\leqslant ns$ in the variables $t = (t^1, ..., t^s)$, and it immediately follows from the Newton formula (4) in one variable, that the difference $f(t) - \mathscr{P}(t)$ vanishes at all points of the Cartesian grid $\mathbf{X} = X^1 \times \cdots \times X^s \subset \mathbb{R}^s$.

Moreover, replacing each X^j by $\operatorname{diag}^m(X^j)$, one obtains the interpolating polynomial that, as a function of t, has the same m-jet as f at every point of the initial grid $\mathbf{X}$. The degree of this polynomial will be $\leqslant ns(m+1)$, and derivatives of it restore the partial derivatives of f at all grid points.

The last remark we make concerns replacing the scalar multivariate function f by a multivariate vector-function $f: \mathbb{R}^s \to \mathbb{R}^k$. Then all divided differences become k-dimensional vectors, $\mathscr{P}(t)$ will become a vector polynomial, but no changes in the construction are required.

Denote by $\mathbf{D}(n, m, s, k)$ the collection of all divided differences with m repetitions, $\{\mathbf{\Delta}_x^\alpha f(\operatorname{diag}^m(X^1), ..., \operatorname{diag}^m(X^s))\}_\alpha$, $\alpha_i \leqslant (m+1)n$, $i = 1, ..., s$. This is a linear space naturally equipped with the coordinates $\{x_i, u_\alpha : 0 \leqslant i \leqslant n, \alpha_i \leqslant (m+1)\, n\}$, where x_i (resp., u_α) are vectors from $\mathbb{R}^s$ (resp., $\mathbb{R}^k$). The dimension of this space is equal to $ns + k \cdot (n(m+1))^s$.

In the same way as before, let $\mathscr{D}^{n,m}f$ be the DD-extension with m repetitions of a smooth function f of s independent variables: by definition, $\mathscr{D}^{n,m}f$ maps $\mathbb{R}^{ns}$ to $\mathbf{D}(n, m, s, k)$. The multivariate interpolation formula together with its derivatives in t_j evaluated at the points of the grid, can be interpreted as a polynomial map restoring multijets from divided differences.

NEWTON INTERPOLATION ON $\mathbb{R}^s$ (ABSTRACT VERSION). *The multivariate Newton interpolation formula* (6) *defines a polynomial interpolation map* π: $\mathbf{D}(n, m, s, k) \to \mathbf{J}(n, m, s, k)$ *such that* $\mathscr{J}^{n,m}f = \pi \circ \mathscr{D}^{n,m}f$. *The degrees of the components of* π *do not exceed* $ns(m+1)$.

Remark. The data stored in the collection of multivariate divided differences is by far more excessive than necessary to restore the multijets: the divided differences $\boldsymbol{\Delta}^{\alpha}_{x} f(\mathrm{diag}^m(X^1), ..., \mathrm{diag}^m(X^s))$ allow for restoring any partial derivative $(\partial^\beta f/\partial x^\beta)(x^1_{\sigma(1)}, ..., x^s_{\sigma(s)})$ for any multi-index $\beta \in \mathbb{Z}^s_+$ with $\beta_j \leqslant m$ and any index map σ: $\{1, 2, ..., s\} \to \{1, 2, ..., n\}$. The multijet space corresponds only to the derivatives with $|\beta| \leqslant m$ and the constant maps $\sigma_i \equiv i$, $i = 1, ..., n$.

One can in fact reduce the number of divided differences by replacing the Cartesian grid $\mathbf{X}$ by a triangular one (notice that even for $m = 0$ the remainder term of the interpolation formula is s-flat at all points of this grid. This would lead to more cumbersome formulas and computations, but, as a result, one has a better estimate for the degree of the map π: as shown in [G], it can be made not exceeding $(m+1)(n+s)$.

3. DEMONSTRATION OF THE MAIN THEOREM AND DISCUSSION

3.1. *Transversality Theorem in the DD-spaces*

The main reason why the multijet space has to be replaced by the divided difference space is that by small smooth variations of f one can achieve transversality to everything in the latter (but not in the former) space. More precisely, we have the following simple observation that generalizes (to a certain extent) the strong form of the Thom transversality theorem [GG] and is proved by similar arguments.

TRANSVERSALITY THEOREM. *Let* $S \subset \mathbf{D}(n, m, s, k)$ *be any stratified subvariety of the divided difference space* (*e.g., a real algebraic subset*). *Then the maps whose DD-extensions with m repetitions are transversal to S, constitute a residual subset in the space* $C^{n(m+1)s}(\mathbb{R}^s, \mathbb{R}^k)$.

Proof. The proof immediately follows from the general result from [GG] and the submersivity of the perturbed polynomial family

$$F(\cdot, \varepsilon) = \mathscr{D}^{n,m}\left(f(x) + \sum_A \varepsilon_\alpha x^\alpha\right), \qquad A = \{\alpha \in \mathbb{Z}^s_+ : 0 \leqslant \alpha_i \leqslant n(m+1) - 1\}$$

More precisely, consider first the simplest case $k = 1$ (scalar multivariate divided differences). We claim that the map taking the collection of points $(x_1, ..., x_n) \in (\mathbb{R}^s)^n$ and the parameters $\{\varepsilon_\alpha\}$ into the complete collection of divided differences with repetitions, has the full rank.

This is almost obvious. Let (x_i, u_α) be the canonical coordinates on the DD-space. Then the partial derivative $\partial u_\alpha / \partial \varepsilon_\beta$ is the DD-extension $\mathbf{\Delta}^\alpha_x x^\beta$ of the *monomial function* x^β, and one can easily see that it is equal to 1 (constant) if $\alpha = \beta$ and vanishes if $\alpha_j > \beta_j$ for at least one $j = 1, ..., s$. This means that the matrix of partial derivatives of the map F is "upper-triangular" with respect to the natural ordering of the multiindices ($\alpha \prec \beta \Leftrightarrow |\alpha| < |\beta|$). This "triangularity" immediately implies the nondegeneracy. It remains only to observe that, by [GG], a submersive family is transversal to all submanifolds (strata of S) for a generic value of the parameters ε.

The case of vector-valued functions is analyzed using the the same arguments verbatim, with the only change: ε_α should be introduced as k-dimensional vector rather than scalar parameters. ∎

3.2. *Proof of the Main Theorem*

Let $\Sigma \subset \mathbf{J}(n, m, s, k)$ be an algebraic subset given by polynomial equations of degrees $\leqslant d$. Then the preimage $S = \pi^{-1}(\Sigma) \subset \mathbf{D}(n, m, s, k)$ is an algebraic subvariety in the divided difference space, defined by polynomials of degree $\leqslant n(m+1)\, sd$. As already mentioned in Section 1.3, the set S can be stratified, and its contiguity number does not exceed $2(n(m+1)\, sd)^{ns + k \cdot (n(m+1))^s}$. This completes the proof, since the contiguity number of the transversal preimage cannot exceed that of the image [IY]. ∎

In fact, the assumption on algebraicity is too strong. The following definition absorbs all requirements for the above proof to work in the non-algebraic case.

Definition. A submanifold Σ in the multijet space is called *matching the diagonal*, if its preimage $\pi^{-1}(\Sigma)$ is a stratified submanifold in the corresponding divided difference space.

Of course, any algebraic submanifold nicely matches the diagonal, since the blow-up π is polynomial. The same holds also real analytic subvarieties.

General Stratifiability of Multijet Preimages. *The multijet preimage of a submanifold nicely matching the diagonal in a multijet space, is generically a stratified subvariety.*

3.3. *Examples*

Now we can return to the example discussed in the introduction. The variety $\Sigma_\lambda = \{\lambda_1 v_1 + \lambda_2 v_2 = \lambda_0\}$ corresponding to the locus $\{\lambda_1 v_1 + \lambda_2 v_2 = \lambda_0\}$ after blowing-up $v_1 = u_0$, $v_2 = u_0 + u_1(x_2 - x_1)$ is transformed into the surface $\pi^{-1}(\Sigma)$ that is a *nonsingular* algebraic surface in the DD-space (x_1, x_2, u_0, u_1) for all parameters $(\lambda_0 : \lambda_1 : \lambda_2)$ different from $(0 : 1 : -1)$.

For the latter value the surface becomes *singular*: $\pi^{-1}(\Sigma_{(0\,:\,-1\,:\,1)}) = \{u_1(x_2 - x_1) = 0\}$ is the union of two transversal hyperplanes. This union is obviously a stratified variety with the contiguity number equal to 4, hence its typical preimage is stratified with the same contiguity number. This agrees with the conclusion achieved by elementary considerations in the introduction.

We conclude by an example of a smooth manifold that is *not* nicely matching the diagonal. In the same space with the coordinates (x_1, x_2, v_1, v_2) consider the locus $\{v_2 - v_1 = \varphi(v_1)\}$, where $\varphi(\cdot) \not\equiv 0$ is a C^∞-smooth function of one variable having an infinite number of accumulating zeros. The blow-up of this locus is the set $\{u_1(x_2 - x_1) = \varphi(u_0)\}$ that is not stratifiable (cannot be represented as a locally finite union of strata near the point of accumulation of zeros). Therefore the original locus is not matching nicely the diagonal.

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REFERENCES

[BZ] I. S. Berezin and N. P. Zhidkov, "Computing Methods," Vol. 1, Pergamon, Oxford, UK, 1965.

[GG] M. Golubitski and V. Guillemin, "Stable Mappings and their Singularities," Springer-Verlag, Berlin/Heidelberg/New York, 1973.

[G] A. Grigoriev, "The Local Topological Structure of Generic Multijet Preimages of Algebraic Varieties," M.Sc. Thesis, Weizmann Institute, Rehovot, 1996.

[H] J. Heintz, Definability and fast quantifier elimination in algebraically closed fields, *Theoret. Comput. Sci.* **24** (1983), 239–277.

[IY] Yu. Ilyashenko and S. Yakovenko, Finite cyclicity of elementary polycycles in generic families, *in* "Concerning the Hilbert 16th Problem," Amer. Math. Soc. Transl. Ser. 2, Vol. 165, pp. 21–95, Amer. Math. Soc., Providence.

[K] V. Kaloshin, Bifurcation of periodic orbits from spatial polycycles, Princeton University, preprint, 1997.

[M] J. N. Mather, Stratifications and mappings, *in* "Dynamical Systems" (M. Peixoto, Ed.), pp. 195–232, Academic Press, San Diego, 1973.

[W] H. Whitney, Elementary Structure of real algebraic varieties, *Ann. Math.* **66** (1957), 545–556.